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Simple Examples with Features of Renormalization for Turbulent Transport

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Simple examples with features of renormalization for turbulent transport

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Two simple exactly solvable models for turbulent transport are introduced and discussed here with complete mathematical rigour. These models illustrate several different facets of super-diffusion and renormalization for turbulent transport. The first model involves time dependent velocity fields with suitable long-range correlations and the complete renormalization theory is developed here in detail. In addition rigorous examples are developed by using variants of this model where the effective equation for the ensemble average at large scales and long times is diffusive despite the fact that each realization exhibits catastrophic large-scale instability. The second model introduced previously by the authors involves transport-diffusion in simple shear layers with turbulent velocity statistics. The theories of renormalized eddy diffusivity and higher-order statistics are surveyed here. An extreme limiting case of the theory involving turbulent velocity statistics with long-range spatial correlations but gaussian white noise in time is discussed in detail. Both the renormalized theory of eddy diffusivity and exact explicit equations for second-order correlations related to the pair distance function are developed in complete detail here in this instructive limiting case.

1. Introduction

The advection-diffusion of a passive scalar by an incompressible velocity is described by the equation

$$\partial T/\partial t + (\boldsymbol{v}\cdot\nabla)\,T = \kappa\Delta T, \tag{1.1}$$

where the incompressible velocity field, $\boldsymbol{v}(x,t)$, satisfies $\operatorname{div}\boldsymbol{v} = 0$ and $\kappa \geqslant 0$ is the diffusion coefficient. The problem in (1.1) is especially important and difficult when the velocity field v involves a wide range of excited space and/or time scales and admits a statistical description. Practical applications where these are the circumstances include predicting temperature profiles in high Reynolds number turbulence (Batchelor 1982), the tracking of pollutants in the atmosphere (Csanady 1973), and the diffusion of tracers in heterogeneous porous media (Dagan 1987). Besides the practical interest in the equation from (1.1), the behaviour of solutions of (1.1) with statistical velocity fields is an important prototype problem for turbulence theories involving the Navier–Stokes equations (McComb 1990) since the equation in (1.1) is statistically nonlinear even though this equation is linear for a given realization. Thus the problem in (1.1) with velocity fields which are statistical

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with a wide range of scales has been attacked through a variety of physical space and Fourier space renormalization theories which typically utilize partial summation of divergent perturbation series according to various recipes (Kraichnan 1965, 1970, 1987; Roberts 1961; Forster *et al.* 1977; Rose 1977; Yakhot & Orszag 1986, 1988; Koch & Brady 1989).

Statistical quantities of physical interest include the mean concentration, $\langle T\rangle$, and the second-order correlations, $\langle T(x,t)\,T(x',t)\rangle$ which are related directly to the relative diffusion of pairs of particles (Lesieur 1990, ch. 8) as well as more complex higher-order statistics involving the fractal dimension of level sets and/or interfaces. One of the important practical problems pioneered by G. I. Taylor (1953) involves developing effective approximate equations at large scales and long times for the mean concentration, $\langle T\rangle$; in the context of (1.1) these issues are also significant prototype problems for theories of eddy diffusivity for the Navier–Stokes equations (Kraichnan 1987; Yakhot & Orszag 1986). The solution of these problems has practical implications for numerical simulation on the largest contemporary supercomputers in disciplines such as atmospheric science because simplified effective equations are needed to assess the effects of the continuum of energetic but unresolved turbulent scales of motion on the larger scales without calculating these effects explicitly. As pioneered by Richardson (Richardson 1926; Batchelor 1951) the relative diffusion of particle pairs is crucial for estimating, for example, the size of clouds of pollutants (Csanady 1973) or mixing processes in turbulent combustion (Borghi 1988). All of these problems are especially difficult because velocity fields with spectra such as the Kolmogoroff spectrum involve infrared divergences and no separation of scales (Yakhot & Orszag 1986; McComb 1990) so conventional diffusion theories can fail for these physically important cases.

Recently the authors have introduced a family of simple model problems with turbulent velocity statistics where the above issues can be understood and clarified in an unambiguous and mathematically rigorous fashion (Avellaneda & Majda 1990*a*, *b*, 1992*a–d*; Majda 1991). These simple models have an exactly solvable renormalization theory for statistical behaviour at large scales and long times and exhibit various 'phase transitions' from conventional diffusion theories with scale-separated velocity statistics to more complex equations involving anomalous super-diffusion as the velocity statistics vary to allow infrared divergences and long-range correlation without separation of scales. The models are the special case of (1.1) involving simple shear flows with turbulent velocity statistics. Thus, the model problem has the form

$$\partial T/\partial t + v(x,t)\,\partial T/\partial y = \kappa\,\Delta T. \tag{1.2}$$

The structure of the velocity statistics for (1.2) is summarized briefly in §3 below. The complete theory for renormalized eddy diffusivity for (1.2) has been developed by the authors (Avellaneda & Majda 1990*a*) and this rigorous theory in a simple model provides an unambiguous test for the capability of renormalization group (RNG) methods and renormalized perturbation theory (RPG) to produce good approximations to these exact results in regions with super-diffusion and infrared divergences; such use of the model as a test problem for RNG and RPG methods has been developed recently (Avellaneda & Majda 1992*b*).

Also the authors have studied more subtle aspects of the models in (1.2) including higher-order statistics such as the relative diffusion of pairs of particles, the sweeping effect of large scales, and the fractal dimension of interfaces. Despite the simplicity of the model in (1.2), the authors have demonstrated rigorously (Avellaneda & Majda

1992*a*) that the model contains, in the vicinity of the Kolmogoroff value, a remarkable amount of the qualitative behaviour of turbulent transport which has been uncovered in recent experiments and proposed phenomenological theories including the Richardson 4/3 law for pair dispersion and fractal dimensions of turbulent interfaces in excellent agreement with recent experimental values. All of these remarkable features of the simplified model problem have led the authors to suggest with analytic supporting evidence (Avellaneda & Majda 1992*c*) that similar scaling behaviour occurs for turbulent transport diffusion by general isotropic incompressible velocity fields in three dimensions with velocity spectra in the vicinity of the Kolmogoroff spectrum. Other aspects of the simplified model involving statistical universality and also general perturbations of (1.2) to slightly stratified flow have been developed recently (Avellaneda & Majda 1992*a*, *d*) as well as a different viewpoint for large scale, long time diffusion for (1.2) with steady velocity fields with infrared divergences involving rigorous diagrammatic perturbation theory (Avellaneda & Majda 1990*b*). It is worth mentioning here that the models in (1.2) are a generalization to turbulent velocity statistics with infrared divergence and no separated scales of the exactly solvable model illustrating enhanced diffusion developed by Taylor (1953). However, the methods of exact solution are quite different in these turbulent regimes and involve careful asymptotic evaluation of function-space integrals through the Feynman–Kac formula (Avellaneda & Majda 1990*a*, 1992*a*).

Here we study several facets of super-diffusion and renormalization in turbulent transport which can be illustrated in simplified model problems. In §2 we give a complete treatment of super-diffusion and renormalization for a class of models of (1.1) with the special form,

$$\partial T/\partial t + \boldsymbol{v}(t)\cdot\nabla T = \kappa\,\Delta T, \tag{1.3}$$

where $v(t)$ is a random velocity field. Such simple model problems were introduced by Kubo (1963) in a basic paper. When the correlations for $v(t)$ are sufficiently short range the well-known Kubo theory is valid; however, if the velocity field exhibits sufficiently long-range correlations, a 'phase transition' occurs together with super-diffusion and a different structure for the effective diffusion equation for $\langle T\rangle$ at large scales and long times. The examples in (1.3) are the simplest ones known to the authors which exhibit this renormalized behaviour and a complete mathematical treatment as well as other interesting facets of (1.3) and related equations are presented in §2.

In §3 several aspects of the exact renormalization theory for (1.2) which have been developed by the authors are discussed in general and illustrated in detail for a special class of velocity statistics which are an extreme limiting case of those considered in our earlier work; these velocity statistics involve gaussian white noise decorrelation in time at all wave numbers but long range spatial correlations and thus have the opposite character from the velocity statistics used for (1.3) in §2. We give a self-contained treatment of the 'phase diagram' for (1.2) with these special statistics as well as the theory of renormalized eddy diffusivity and explicit equations for the pair-distance function $\langle T(x,y,t)\,T(0,0,t)\rangle$ in this limiting case.

2. Renormalization in the simplest model

Here we consider the statistical averaging of the model equation

$$\begin{aligned} \partial T/\partial t + \boldsymbol{v}(t)\cdot\nabla T &= \kappa\,\Delta T, \quad x\in R^d, \quad t>0, \\ T(x,0) &= T_0(x). \end{aligned} \tag{2.1}$$

The constant, $\kappa \geqslant 0$, is the bare (molecular) diffusivity. We assume that $v(t) = (v_1(t), \ldots, v_d(t))$ is a stationary mean-zero gaussian random field (Yaglom 1962) with symmetric correlation matrix $R(t) = (R_{ij}(t))$ where

$$R_{ij}(|t|) = \langle v_i(t+\tau)\, v_j(\tau)\rangle. \tag{2.2}$$

Here and elsewhere in this paper, $\langle\cdot\rangle$ denotes the ensemble average over velocity statistics. In this section, we assume for simplicity in exposition that the initial data, $T_0(x)$, is a prescribed smooth deterministic function with a Fourier transform of compact support.

For any given realization of the velocity field, the problem in (2.1) is readily solved exactly through spatial Fourier transforms. With

$$f(x) = \int_{R^d} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} \hat{f}(\xi)\,\mathrm{d}\xi \tag{2.3}$$

the solution of (2.1) for a given realization of the velocity field is given by

$$T(x,t) = \int \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,\mathrm{e}^{-4\pi^2\kappa|\xi|^2 t} \exp\left[-2\pi\mathrm{i}\int_0^t \boldsymbol{v}(s)\cdot\boldsymbol{\xi}\,\mathrm{d}s\right] \hat{T}_0(\xi)\,\mathrm{d}\xi. \tag{2.4}$$

The only random function that occurs in (2.4) is

$$\exp\left[-2\pi\mathrm{i}\int_0^t \boldsymbol{v}(s)\cdot\boldsymbol{\xi}\,\mathrm{d}s\right].$$

For velocity fields with gaussian statistics, we will use the following principle in averaging (2.4) (Gelfand & Vilenkin 1964):

If $v(t)$ is a mean-zero stationary gaussian process, then

$$\left\langle \exp\left[-\mathrm{i}\int_0^t \boldsymbol{v}(s)\cdot\boldsymbol{\xi}\,\mathrm{d}s\right]\right\rangle = \exp\left[-\frac{1}{2}\left(\sum_{i,j=1}^{d} \xi_i\,\xi_j \int_0^t\int_0^t R_{ij}(|s-s'|)\,\mathrm{d}s\,\mathrm{d}s'\right)\right]. \tag{2.5}$$

Below we also use the elementary identity,

$$\frac{1}{2}\int_0^t\int_0^t R_{ij}(|s-s'|)\,\mathrm{d}s\,\mathrm{d}s' = \int_0^t (t-s)\,R_{ij}(|s|)\,\mathrm{d}s. \tag{2.6}$$

We remark that (2.5) can be proved in an elementary manner in the following fashion: first, if the integral is replaced by a Riemann sum, (2.5) is an identity that follows from the definition of gaussian statistics; then the dominated convergence theorem can be applied to pass to the limit and obtain (2.5) under the assumption that almost every realization is continuous; this assumption can then be removed by further approximation. The formula in (2.5) is the main fact from probability theory that we need for the remainder of this section.

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(a) Well-posed and ill-posed equations for the mean, $\langle T\rangle$

We consider the mean $\langle T\rangle(x,t)$, and ask whether there is a simple differential equation always satisfied by this mean quantity. With (2.5) and (2.6) we average (2.4) and obtain

$$\langle T\rangle = \int \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,\mathrm{e}^{-4\pi^2\xi^2\kappa t}\exp\left(-4\pi^2\sum_{i,j=1}^{d}\xi_i\,\xi_j\int_0^t (t-s)\,R_{ij}(s)\,\mathrm{d}s\right)\hat{T}_0(\xi)\,\mathrm{d}\xi. \tag{2.7}$$

From (2.7) we observe that $\langle T\rangle$ satisfies the differential equation,

$$\frac{\partial}{\partial t}\langle T\rangle = \kappa\,\Delta\langle T\rangle + \sum_{i,j=1}^{d} D_{ij}(t)\,\frac{\partial^2}{\partial x_i\,\partial x_j}\langle T\rangle, \quad \langle T\rangle|_{t=0} = T_0(x), \tag{2.8}$$

with
$$D_{ij}(t) = \int_0^t \langle v_i(s)\,v_j(0)\rangle\,\mathrm{d}s. \tag{2.9}$$

If the equation in (2.8) is to be useful in any practical sense, it should define a well-posed problem for any starting time $t_0 > 0$. This requirement is needed to avoid, for example, catastrophically amplifying round off error in a numerical approximation to (2.8). Since $D_{ij}(t)$ in (2.8) is a symmetric matrix, it is not difficult to verify the following necessary and sufficient conditions for (2.8) to define a well-posed initial value problem for any $t_0 \geqslant 0$:

The differential equation in (2.8) defines a well-posed problem if and only if the symmetric matrix $\kappa\delta_{ij} + D_{ij}(t_0)$ is non-negative for any $t_0 \geqslant 0$, i.e.

$$\kappa|\xi|^2 + \sum_{i,j=1}^{d} D_{ij}(t_0)\,\xi_i\,\xi_j \geqslant 0 \quad \text{for all} \quad \xi\in R^d. \tag{2.10}$$

For the special case of (2.1) with $v(t) = (v_1(t), 0, \ldots, 0)$ the condition in (2.10) becomes

$$\kappa + \int_0^{t_0} R(s)\,\mathrm{d}s \geqslant 0, \quad t_0 \geqslant 0 \tag{2.11}$$

with $R(s) = \langle v_1(s)\,v_1(0)\rangle$. Thus, if the velocity $v_1(t)$ has negative correlations over a sufficiently wide range of integration, then the differential equation in (2.8) for the mean, $\langle T\rangle$, is an ill-posed problem. Next we present explicit examples of this phenomenon.

The function $R(t) = (\alpha^2+\beta^2)\,\mathrm{e}^{-\alpha|t|}\cos\beta t$ is the correlation function of a stationary gaussian random field for $\alpha > 0$, $\beta > 0$ (Yaglom 1962, p. 71) and

$$\int_0^{t_0} R(s)\,\mathrm{d}s = \alpha + \mathrm{e}^{-\alpha t_0}(-\alpha\cos\beta t_0 + \beta\sin\beta t_0). \tag{2.12}$$

By fixing κ and α but choosing β to be large enough, it is clear that (2.11) will be violated over several intervals in time centred about the points $t_k^* = (2\pi k - \frac{1}{2}\pi)/\beta$ for $k = 1, \ldots, l$ so the averaged equation is ill-posed. More physical examples which can yield an ill-posed averaged equation arise by determining $v(t)$ as the solution of the damped and driven harmonic oscillator

$$\frac{\mathrm{d}^2 v_1(t)}{\mathrm{d}t^2} + 2\alpha \frac{\mathrm{d}v_1(t)}{\mathrm{d}t} + \omega^2 v_1(t) = A\,W(t), \tag{2.13}$$

where $W(t)$ is stationary gaussian white-noise so that $\langle W(t), W(t')\rangle = \delta(t-t')\,\mathrm{d}t$. In this example the correlation function is given by (Yaglom 1962, pp. 74, 75)

$$R(t) = (A/2\alpha\omega^2)\,\mathrm{e}^{-2\pi\alpha|t|}(\cos(2\pi\beta t) + (\alpha/\beta)\sin 2\pi\beta|t|), \tag{2.14}$$

where $\beta^2 = \omega^2 - \alpha^2 > 0$ and we have assumed that $\omega^2 > \alpha^2$. With the similarity between the formula in (2.14) and the formula for $R(t)$ used in (2.12), it should be clear to the reader that the equation for $\langle T\rangle$ is ill-posed for a fixed κ with the velocity determined by (2.13) provided the oscillation frequency ω is much larger than the damping rate α in the same fashion as we discussed for the earlier example.

(b) *Coarse graining and Kubo's diffusivity*

Kubo used the model in (2.1) with $\kappa = 0$ in his pioneering paper (Kubo 1963) to illustrate, among other results, the fact that the effect of randomness on the mean is always enhanced diffusion provided the problem is viewed at large scales and long times, i.e. coarse grained, and the correlation functions $R_{ij}(t)$ decay sufficiently rapidly. Thus, the somewhat surprising effects of randomness in creating ill-posed problems described in §2*a* all disappear and the problem becomes well-posed with enhanced diffusion after suitable coarse graining. Here in a straightforward fashion we give precise necessary and sufficient conditions in the model for Kubo's theory to apply.

We consider the problem in (2.1) with the large scale initial data,

$$T^{\delta}(x,0) = T_0(\delta x), \quad \delta \ll 1 \tag{2.15}$$

and introduce the large-scale variables with diffusive scaling given by

$$x' = \delta x, \quad t' = \delta^2 t. \tag{2.16}$$

The coarse-grained mean, $\bar{T}(x',t')$ is defined by

$$\bar{T}(x',t') = \lim_{\delta\to 0}\left\langle T^{\delta}\left(\frac{x'}{\delta}, \frac{t'}{\delta^2}\right)\right\rangle \tag{2.17}$$

with $T^{\delta}(x,t)$ given by the solution of (2.1) with the initial data in (2.15). Using (2.4)–(2.6) and dropping the primes in (2.17) we calculate that

$$\left\langle T^{\delta}\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right)\right\rangle = \int_{R^d} \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,\mathrm{e}^{-4\pi^2\kappa|\xi|^2 t}$$
$$\times\exp\left(-4\pi^2\left[\sum_{j,k}\xi_j\,\xi_k\left(t\int_0^{t/\delta^2} R_{jk}(s)\,\mathrm{d}s - \delta^2\int_0^{t/\delta^2} sR_{jk}(s)\,\mathrm{d}s\right)\right]\right)\hat{T}_0(\xi)\,\mathrm{d}\xi. \tag{2.18}$$

Two conditions are needed for (2.18) to have a limit as $\delta\to 0$ and both requirements involve sufficient decay of the correlation functions $R_{jk}(t)$ as $t\nearrow\infty$; these two conditions are

$$\int_0^{\infty} R_{jk}(s)\,\mathrm{d}s < \infty \quad \text{and} \quad \lim_{T\to\infty}\left(\frac{1}{T}\int_0^{T} sR_{jk}(s)\,\mathrm{d}s\right) = 0. \tag{2.19}$$

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With (2.18) and (2.19) it is clear that the large-scale long-time limit of the mean, $\bar{T}(x,t) = \lim_{\delta\to 0}\langle T^{\delta}(x/\delta, t/\delta^2)\rangle$, exists and satisfies the equation,

$$\frac{\partial \bar{T}}{\partial t} = \kappa\,\Delta\bar{T} + \sum_{i,j=1}^{d} D^{*}_{ij}\frac{\partial^2\bar{T}}{\partial x_i\,\partial x_j}, \quad \bar{T}(x,0) = T_0(x), \tag{2.20}$$

with

$$D^{*}_{ij} = \frac{1}{2}\int_{-\infty}^{\infty} R_{ij}(s)\,\mathrm{d}s. \tag{2.21}$$

We claim that the coarse-grained problem in (2.20) is always well-posed with enhanced diffusion due to the randomness because

$$\sum_{i,j=1}^{d} D^{*}_{ij}\,\xi_i\,\xi_j \geqslant 0 \quad \text{for all} \quad \xi\in R^d. \tag{2.22}$$

From the Bochner–Khinchin theorem (Yaglom 1962, p. 47), the Fourier transform of a correlation matrix is a positive semi-definite measure and $\int_{-\infty}^{\infty} R_{ij}(s)\,\mathrm{d}s$ is simply the Lebesgue density of this measure at the origin so that (2.22) is satisfied automatically. The two examples discussed in (2.12)–(2.14) from §2*a* clearly satisfy the requirements in (2.19) and generate enhanced diffusion at large scales and long times even though the averaged equations are ill-posed without the coarse graining procedure.

There is a large literature which generalizes the Kubo theory to nonlinear random transport operators under suitable and more stringent conditions on decay of the correlation functions beyond those in (2.19) and results in elegant nonlinear versions of the central limit theorem (Khas'minskii 1966, 1980; Papanicolaou & Kohler 1974).

(i) *Remark*

If general initial data $T_0(x)$ is utilized rather than large scale initial data and the coarse graining procedure from (2.16) and (2.18), (2.19) is repeated, the scaled mean $\bar{T}(x,t) = \lim_{\delta\to 0}\delta^{-d}\langle T^{\delta}(x/\delta, t/\delta^2)\rangle$ exists and satisfies the same equation in (2.20) with the initial data, $\bar{T}(x,t)|_{t=0} = C_0\,\delta(x)$ provided that $C_0 = \int T_0\,\mathrm{d}x \neq 0$. The solution is simply a multiple of the Green's function for (2.20). If $C_0 = 0$ but other higher moments of T_0 are non-zero, then similar arguments apply.

(*c*) *Renormalization and super-diffusion with long-range correlations*

We begin by introducing a family of gaussian velocity statistics depending on a parameter ϵ. As the parameter ϵ increases longer-range correlations in the velocity statistics build up in time. The reason why we do this involves the analogy with Wilson's theory of critical phenomena in statistical physics (Ma 1976); in that work the role of ϵ was played by the parameter $4-d$ with d the space dimension and for dimensions $d>4$ (simple) mean field theory applies while for dimensions $d<4$ new anomalous scaling phenomena occur. The use of spectral parameters which vary while the dimension d is kept fixed to generate analogues of Wilson's theory for Navier–Stokes turbulence at large scales and long times is one of the main approaches in renormalization theory (Forster *et al.* 1977), and often involves perturbation expansions in ϵ (Yakhot & Orszag 1986). Such parametrized families of velocity statistics have been used by the authors in the renormalization theory for the model shear layer problem from (1.2) (Avellaneda & Majda 1990*a*, 1992*a*, *b*; Majda 1991) and will be described briefly below in §3. In the renormalization theory

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for the simple model discussed here in §2, the analogue of mean field theory from critical phenomena (Ma 1976) is the simple Kubo theory described in §2*b*; we parametrize the velocity statistics so that the mean field theory occurs in the model from (2.1) with $\epsilon < 0$ while anomalous phenomena requiring renormalization occur for $\epsilon > 0$.

With the above discussion as motivation, we consider gaussian velocity statistics with a power spectrum given by

$$|\omega|^{-\epsilon}\varphi_\infty(|\omega|) \quad \text{for } -\infty < \epsilon < 1, \tag{2.23}$$

where $\varphi_\infty(s)$ is a smooth rapidly decreasing even function of s with φ_∞ identically one in a neighbourhood of the origin and $\varphi_\infty(s) \geqslant 0$. We choose $\bar{R}_{ij}$ to be a fixed non-zero positive symmetric matrix so that

$$\sum_{i,j=1}^{d} \bar{R}_{ij}\,\xi_i\,\xi_j \geqslant 0. \tag{2.24}$$

With the restrictions, $-\infty < \epsilon < 1$, it follows from (2.23), (2.24) and the Bochner–Khinchin theorem that a stationary mean zero gaussian random field $v^\epsilon(t)$ exists with velocity correlations $R^\epsilon_{ij}(t) = \langle v^\epsilon_i(t+\tau)\, v^\epsilon_j(\tau)\rangle$ given by

$$R^\epsilon_{ij}(t) = \bar{R}_{ij}\int \mathrm{e}^{2\pi \mathrm{i}\omega t}\,|\omega|^{-\epsilon}\varphi_\infty(|\omega|)\,\mathrm{d}\omega \tag{2.25}$$

for $-\infty < \epsilon < 1$. The following elementary proposition gives the large time behaviour of the correlation functions in (2.25) as a function of the parameter ϵ.

Proposition 1. *The correlation functions in* (2.25) *are smooth functions of t with the following behaviour*:

(*a*)

$$|R^\epsilon_{ij}(t)| \leqslant C^\epsilon(1+|t|)^{-1+\epsilon}; \tag{2.26}$$

(*b*) *for* $0 < \epsilon < 1$ *and* $|t| \geqslant 1$, *the correlation functions have the asymptotic expansion*

$$R^\epsilon_{ij}(t) = \bar{R}_{ij}\,A_\epsilon|t|^{-1+\epsilon} + E^\epsilon(t) \tag{2.27}$$

with $A_\epsilon = (2\pi)^\epsilon \sin(\tfrac{1}{2}\epsilon\pi)\,\Gamma(-\epsilon+1)$ *and* $E^\epsilon(t)$ *satisfying*

$$|E^\epsilon(t)| \leqslant C_N(1+|t|)^{-N} \tag{2.28}$$

for any $N > 0$.

We omit the straightforward proof of this elementary proposition which utilizes scaling and the method of stationary phase in its simplest form.

We consider the large-scale long-time behaviour of the mean statistics for the model problem

$$\left.\begin{aligned} \partial T^\delta/\partial t + \boldsymbol{v}^\epsilon(t)\cdot\nabla T^\delta &= \kappa\,\Delta T^\delta,\\ T^\delta|_{t=0} &= T_0(\delta x), \quad \delta \ll 1 \end{aligned}\right\} \tag{2.29}$$

as ϵ varies for $-\infty < \epsilon < 1$. First, for $\epsilon \leqslant 0$ (2.26) guarantees that the conditions in (2.19) for the Kubo theory §2*b* are satisfied so that with the simple diffusion scaling in (2.16) the averaged equation for the coarse-grained limit is given by (2.20); in fact, it is a simple matter to calculate from (2.23) that the enhanced diffusion, D^*_{ij}, vanishes for $\epsilon < 0$.

For $\epsilon > 0$, the asymptotic behaviour of the correlation function in (2.27) guarantees that the correlations decay slowly so the integrals in (2.19) diverge and the simple

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Kubo theory from §2*b* is no longer valid. Thus the problem needs to be renormalized on a different timescale. We introduce the new large-scale long-time scaling functions

$$x' = \delta x, \quad t' = \rho^2(\delta)\, t \tag{2.30}$$

and attempt to choose $\rho(\delta)$ to renormalize this divergence. Dropping the primes in (2.30) and by following (2.6) and the same procedure from §2*b* we calculate that

$$\left\langle T\left(\frac{x}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle = \int_{R^d} \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\, \hat{T}_0(\xi)\, \mathrm{e}^{-4\pi^2(\delta^2/\rho^2)\,\kappa t}$$
$$\times \exp\left(-4\pi^2\left[\sum_{jk} \xi_j\,\xi_k\left(\frac{\delta^2}{\rho^2} t \int_0^{t/\rho^2} R^{\epsilon}_{jk}(s)\,\mathrm{d}s - \delta^2 \int_0^{t/\rho^2} sR^{\epsilon}_{jk}(s)\,\mathrm{d}s\right)\right]\right)\mathrm{d}\xi. \tag{2.31}$$

We use the asymptotic expansion from (2.27) and (2.28) and compute that

$$\frac{\delta^2}{\rho^2} t \int_0^{t/\rho^2} R^{\epsilon}_{jk}(s)\,\mathrm{d}s - \delta^2 \int_0^{t/\rho^2} sR^{\epsilon}_{jk}(s)\,\mathrm{d}s = \frac{\delta^2}{\rho^{2+2\epsilon}}\frac{A_\epsilon\, t^{1+\epsilon}}{\epsilon(\epsilon+1)}\bar{R}_{jk} + O\left(\frac{\delta^2}{\rho^2}\right). \tag{2.32}$$

We introduce the limiting coarse-grained mean

$$\bar{T}(x,t) = \lim_{\delta\to 0}\left\langle T^{\delta}\left(\frac{x}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle \tag{2.33}$$

and require that $\bar{T}$ has non-trivial dynamical behaviour, i.e. $\bar{T}$ is neither identically zero nor merely the initial data $T_0(x)$. With (2.31) and (2.32), we see that such non-trivial limiting behaviour occurs for a unique choice of the scaling function $\rho(\delta)$ determined by (2.32), i.e.

$$\rho(\delta) = \delta^{1/(1+\epsilon)}, \quad 0 < \epsilon < 1 \tag{2.34}$$

and with this choice $\delta^2/\rho^2 \to 0$ so the bare diffusivity κ is irrelevant in this régime and

$$\bar{T}(x,t) = \int_{R^d} \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\, \hat{T}_0(\xi) \exp\left(-4\pi^2\left[\sum_{j,k} \xi_j\,\xi_k\,\bar{R}_{jk}\frac{A_\epsilon\, t^{1+\epsilon}}{\epsilon(\epsilon+1)}\right]\right)\mathrm{d}\xi. \tag{2.35}$$

Thus, the large-scale long-time renormalized equation for the mean is given by

$$\frac{\partial\bar{T}}{\partial t} = \epsilon^{-1} A_\epsilon\, t^{\epsilon} \sum \bar{R}_{jk}\frac{\partial^2\bar{T}}{\partial x_j\,\partial x_k}, \quad \bar{T}|_{t=0} = T_0(x) \tag{2.36}$$

for $0 < \epsilon < 1$.

We emphasize here that the simple renormalization theory in (2.35) and (2.36) involves super-diffusion. First, the scaling exponent $\rho(\delta) = \delta^{1/(1+\epsilon)}$ for $0 < \epsilon < 1$ involves shorter renormalized times in (2.30) than the standard diffusive timescales for the Kubo theory from (2.16) where $\rho(\delta) = \delta$. Furthermore, on these shorter renormalized timescales particles spread more rapidly. From (2.36) we compute that the second moments of $\bar{T}$ in time are given by

$$\left(\int x_j\, x_k\, \bar{T}\right)(t) = 2\bar{R}_{jk}\frac{A_\epsilon}{\epsilon - \epsilon^2} t^{1+\epsilon} \int T_0(x)\,\mathrm{d}x + \int x_j\, x_k\, T_0(x)\,\mathrm{d}x. \tag{2.37}$$

Thus, with $T_0(x) = \delta(x)$ particles spread at the super-diffusive rate,

$$\int |x|^2\, \bar{T}(t) = 2\left(\sum_j \bar{R}_{jj}\right)\frac{A_\epsilon}{\epsilon - \epsilon^2} t^{1+\epsilon} \tag{2.38}$$

for $0 < \epsilon < 1$ in contrast to the standard spreading $\int |x|^2 \bar{T} \sim Dt$ associated with conventional diffusion in the mean field régime from §2*b* and described by (2.20).

To pursue the analogy with critical phenomena in this simple model, we see that a 'phase transition' occurs from normal diffusion to super-diffusion as ϵ crosses through zero with different effective equations at large scales and long times. Furthermore, the exponent of the time rescaling function $\rho(\delta)$ from (2.34) plays the role of an order parameter characterizing this 'phase transition' since with $f(\epsilon) = \log(\rho(\delta))/\log(\delta)$

$$f(\epsilon) = \begin{cases} 1 & \epsilon \leqslant 0, \\ 1/(1+\epsilon) & 0 < \epsilon < 1, \end{cases} \tag{2.39}$$

and the graph of $f(\epsilon)$ reveals the classical behaviour of an order parameter in a first-order phase transition (Ma 1976). The determination of this large-scale rescaling function $\rho(\delta)$ as 'phase transitions' occur is one of the goals of renormalized theories for eddy diffusivity (Yakhot & Orszag 1986; Avellaneda & Majda 1990*a*, 1992*b*). Another facet of the renormalization theory just presented is that the effective renormalized equation in (2.36) is invariant under the space-time symmetry group associated with (2.30), i.e. solutions of (2.36) are invariant under the transformations

$$(x, t) \to (\lambda x, \lambda^{2/(1+\epsilon)} t) \tag{2.40}$$

for any $\lambda > 0$.

(*d*) *The stabilizing effect of randomness on large scale instability*

The Kuramoto–Sivashinski equation is linearly unstable at large scales with nonlinear energy transfer. By using renormalization group ideas (Forster *et al.* 1977), Yakhot predicted that the intrinsic randomness in solutions of these equations would result in a stabilized effective equation, the viscous Burgers equation, with a random force as the large-scale long-time effective equation for the overall dynamics (Yakhot 1981). This intriguing prediction of Yakhot was confirmed through careful numerical simulation by Zaleski who also gave a detailed numerical study of higher-order statistics (Zaleski 1989). Thus the intrinsic randomness in this nonlinear equation overcomes the large-scale instability for large times and is replaced by a stable effective equation with random forcing. Here we utilize the techniques developed earlier in §2*a–c* to provide extremely simple examples of linear equations with large-scale instability where the effect of randomness does not curtail the growth of instabilities for individual realizations but nevertheless the large-scale long-time effective equation for the ensemble average satisfies a well-posed effective equation. Thus the examples presented below are simplified 'cartoon' models for some aspects of the behaviour observed in much more complex nonlinear systems.

We begin our discussion by considering the linearized Kuramoto–Sivashinski equation with large-scale initial data, i.e.

$$\frac{\partial T^\delta}{\partial t} = -\kappa \frac{\partial^2 T^\delta}{\partial x^2} - \frac{\partial^4 T^\delta}{\partial x^4}, \quad T^\delta(x,t)|_{t=0} = T_0(\delta x), \quad \delta \ll 1. \tag{2.41}$$

Here $\kappa > 0$ is a constant; the equation (2.41) has a band of large-scale unstable wave numbers but is stable for short wavelengths. To study (2.41) at large scales and long times we introduce the scaling

$$x' = \delta x, \quad t' = \tilde{\rho}^2(\delta)\, t \tag{2.42}$$

and after dropping the primes in (2.42), we obtain the rescaled equation

$$\frac{\partial T^{\delta}}{\partial t} = -\frac{\delta^2}{\tilde{\rho}^2(\delta)}\kappa\frac{\partial^2 T^{\delta}}{\partial x^2} - \frac{\delta^4}{\tilde{\rho}^2(\delta)}\frac{\partial^4 T^{\delta}}{\partial x^4}, \quad T^{\delta}|_{t=0} = T_0(x). \tag{2.43}$$

It is easy to check that as $\delta \downarrow 0$ there is a unique scaling with a non-trivial limiting behaviour, namely $\tilde{\rho}(\delta) = \delta$, and the large-scale long-time limit equation is the catastrophically unstable backward heat equation,

$$\partial T/\partial t = -\kappa\,\partial^2 T/\partial x^2, \quad T|_{t=0} = T_0(x). \tag{2.44}$$

This result is a manifestation of the large-scale instability mentioned earlier. It is worth remarking here that at the beginning of §2, we made the standing assumption that $T_0(x)$ has a Fourier transform of compact support; such a condition on the initial data is needed to justify the limiting procedure from (2.43) and (2.44) and also to guarantee that the ill-posed equation in (2.44) has a solution with the given initial data for all time. Thus, this requirement is important for the remainder of this section.

With (2.41)–(2.44) as background, we consider the model problem

$$\frac{\partial T^{\delta}}{\partial t} + v(t)\frac{\partial T^{\delta}}{\partial x} = -\kappa\frac{\partial^2 T^{\delta}}{\partial x^2} - \frac{\partial^4 T^{\delta}}{\partial x^4}, \quad T^{\delta}|_{t=0} = T_0(\delta x), \delta \ll 1, \tag{2.45}$$

where $v(t)$ satisfies the assumptions in (2.19) for the validity of the Kubo theory developed earlier in §2*b*. Clearly the effect of the random convection term in (2.45) is merely to introduce phase shifts so the problem in (2.45) exhibits large-scale instability for each given realization. However, if we consider the limit of the ensemble average at large scales and long times

$$\bar{T}(x,t) = \lim_{\delta\to 0}\left\langle T^{\delta}\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right)\right\rangle,$$

then computations as we developed in §2*b* involving Fourier transforms can be repeated and the effective equation for $\bar{T}$ is given by

$$\frac{\partial \bar{T}}{\partial t} = -\kappa\frac{\partial^2 \bar{T}}{\partial x^2} + D^*\frac{\partial^2 \bar{T}}{\partial x^2}, \quad \bar{T}|_{t=0} = T_0(x) \tag{2.46}$$

with $D^* \geqslant 0$ given by $D^* = \frac{1}{2}\int_{-\infty}^{\infty}\langle v(s)\,v(0)\rangle\,\mathrm{d}s$. Clearly for any velocity statistics with $D^* > \kappa$, the effective equation in (2.46) for the ensemble average, $\bar{T}$, at large scales and long times is stable and any randomness at all has a tendency to make the averaged problem more stable.

Other examples with a similar flavour but with more interesting effective equations for $\bar{T}$ can be developed for the model equations

$$\frac{\partial T^{\delta}}{\partial t} + v^{\epsilon}(t)\frac{\partial T^{\delta}}{\partial x} = \kappa\left|\frac{\partial}{\partial x}\right|^{1+\gamma} T^{\delta} - \frac{\partial^4 T^{\delta}}{\partial x^4}, \quad T^{\delta}|_{t=0} = T_0(\delta x), \quad \delta \ll 1, \tag{2.47}$$

where $v^{\epsilon}(t)$ is a stationary mean zero gaussian random field with long range correlations which satisfy the assumptions in (2.25) from §2*c* for $0 < \epsilon < 1$. Here the operator $|\partial/\partial x|^{1+\gamma}$ is defined through the Fourier transform,

$$f \to \hat{f}(\xi) = \int \mathrm{e}^{-2\pi \mathrm{i}\mathbf{x}\cdot\xi} f(x)\,\mathrm{d}x,$$

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via
$$\left|\frac{\partial}{\partial x}\right|^{1+\gamma} f = \int e^{2\pi i x\cdot\xi}\,|2\pi\xi|^{1+\gamma}\hat{f}(\xi)\,d\xi \tag{2.48}$$

and we assume that γ satisfies $0 < \gamma < 1$.

First we briefly consider the deterministic problem

$$\frac{\partial T^\delta}{\partial t} = \kappa\left|\frac{\partial}{\partial x}\right|^{1+\gamma} T^\delta - \frac{\partial^4 T^\delta}{\partial x^4}, \quad T^\delta|_{t=0} = T_0(\delta x) \tag{2.49}$$

in the large scale limit. With the space-time scalings from (2.42) we obtain the rescaled problem

$$\frac{\partial T^\delta}{\partial t} = \kappa\frac{\delta^{1+\gamma}}{(\tilde{\rho}(\delta))^2}\left|\frac{\partial}{\partial x}\right|^{1+\gamma} T^\delta - \frac{\delta^4}{(\tilde{\rho}(\delta))^2}\frac{\partial^4 T^\delta}{\partial x^4}, \quad T^\delta|_{t=0} = T_0(x). \tag{2.50}$$

There is a unique choice of $\tilde{\rho}(\delta)$ so that the large-scale limit equation is non-trivial; this choice is given by $\tilde{\rho}(\delta) = \delta^{(1+\gamma)/2}$ with the corresponding large-scale limit equation given by

$$\frac{\partial \dot{T}}{\partial t} = \kappa\left|\frac{\partial}{\partial x}\right|^{1+\gamma} T, \quad T|_{t=0} = T_0(x). \tag{2.51}$$

The equation in (2.51) is catastrophically unstable at all wave numbers like the backward heat equation (2.44); furthermore, the random shift from (2.47) does not effect the magnitude of the growth rate of this instability for a given realization.

We return to the problem in (2.47) and calculate the effective equation for the ensemble average $\bar{T}$ at large scales and long times. Clearly the most interesting régime occur when ϵ and γ are chosen so that the effects of randomness and large scale instability occur on the same renormalized timescales, otherwise, one or the other effect completely dominates. Equating the time scalings $\tilde{\rho}(\delta)$ above (2.51) and $\rho(\delta)$ in (2.34) from §2*c*, we compute that this condition will be satisfied provided that

$$1+\gamma = 2/(1+\epsilon), \quad \text{for} \quad 0 < \epsilon < 1. \tag{2.52}$$

With $\rho(\delta) = \delta^{1/(1+\epsilon)}$ and the relation in (2.52), we define $\bar{T}$ by

$$\bar{T}(x,t) = \lim_{\delta\to 0}\left\langle T^\delta\left(\frac{x}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle$$

and repeat the analysis from §2*c* with minor modifications to obtain that the effective equation for the ensemble average in the large-scale long-time limit is given by

$$\frac{\partial \bar{T}}{\partial t} = \kappa\left|\frac{\partial}{\partial x}\right|^{\frac{2}{1+\epsilon}} \bar{T} + \bar{A}_\epsilon t^\epsilon \frac{\partial^2 \bar{T}}{\partial x^2}, \quad \bar{T}|_{t=0} = T_0(x) \tag{2.53}$$

with $\bar{A}_\epsilon = A_\epsilon \bar{R}/\epsilon$. The equation in (2.53) exhibits several interesting properties; this equation is well-posed but has an infinite band of unstable modes at $t = 0$ and a finite band of unstable modes for any $t > 0$ that narrows with time and vanishes in the limit as $t \to \infty$. Clearly the constructions and analysis presented here can be generalized to a large class of constant coefficient equations with suitable random coefficients.

3. Renormalization in the shear layer models

Here we discuss the renormalization theory for the simple models:

$$\frac{\partial T}{\partial t}+v(x,t)\frac{\partial T}{\partial y}=\kappa\,\Delta T,\quad T|_{t=0}=T_0(x), \tag{3.1}$$

where $v(x,t)$ is endowed with suitable turbulent velocity statistics (Avellaneda & Majda 1990*a*, 1992*a*). Below, we will discuss the rigorous theory for eddy diffusivity and also the behaviour of higher-order statistics such as the pair distance function for the simple models. Here we will give a new self-contained treatment of these issues for an extreme limiting case of the turbulent velocity statistics involving white noise in time but long-range spatial correlations; this limiting example will provide an illustration of the techniques and ideas utilized in our work on the model (Avellaneda & Majda 1990*a*, 1992*a–d*).

(*a*) *The turbulent velocity statistics*

(i) *The Kolmogoroff spectrum*

To motivate the families of turbulent velocity statistics utilized in the model, it will be convenient to discuss the general problem of advection-diffusion in (1.1) involving velocity fields with statistics consistent with the Kolmogoroff hypothesis (McComb 1990; Lesieur 1990). With L_0 the integral length scale and $\bar{V}$ the typical velocity of the energy containing scales involving non-universal fluid motions, the Reynolds number is given by $Re=\bar{V}L_0/\nu$ with ν the viscosity of the fluid. Turbulent fluid flows with $Re\nearrow\infty$ and properties of turbulent transport which are valid for $Re\gg 1$ are the primary interest here. The Kolmogoroff hypothesis in d-space dimensions (for $d=3$) asserts that there is a well-defined dissipation length scale L_d so that as $Re\nearrow\infty$, the velocity spectrum has the universal form given by

$$\langle|\hat{v}(k)|^2\rangle=C_0\,\bar{\epsilon}^{\frac{2}{3}}|k|^{1-d-\frac{5}{3}} \tag{3.2}$$

for wave numbers k in the range $L_0^{-1}<|k|<L_d^{-1}$. Here $\bar{\epsilon}$ is the mean dissipation rate and C_0 is a universal constant. The random energy spectrum is assumed to vanish for $|k|>L_d^{-1}$ or decay very rapidly. The velocity is not universal and deterministic on scales larger than L_0. We neglect the non-universal mean flow and continue the discussion.

We non-dimensionalize (1.1) by utilizing the dissipation length scale $L_d=(\nu^2/\bar{\epsilon})^{\frac{1}{4}}$ and the dissipation time scale, $t_d=(\nu/\bar{\epsilon})^{\frac{1}{2}}$. Using the relation $\bar{\epsilon}\approx\bar{V}^3/L_0$, in a familiar fashion we obtain that

$$L_d=(Re)^{-\frac{3}{4}}L_0,\quad t_d=(Re)^{-\frac{1}{2}}\bar{t}, \tag{3.3}$$

where $\bar{t}=L_0/\bar{V}$ is the large-scale eddy turnover time. With this non-dimensionalization and the identification $\delta=(Re)^{-\frac{3}{4}}$, the advection-diffusion equation from (1.1) assumes the form

$$\partial T/\partial t+\boldsymbol{v}_\delta\cdot\nabla T=\tilde{\kappa}\,\Delta T, \tag{3.4}$$

where $\tilde{\kappa}=(Pr)^{-1}$ with $Pr=\nu/\kappa$, the Prandtl number. In (3.4) the rescaled velocity field, $v_\delta(x)$ has the energy spectrum

$$\langle|\hat{v}_\delta(k)|^2\rangle=\begin{cases}C_0|k|^{1-d-\frac{5}{3}}, & \delta<|k|<1,\\ 0, & \text{otherwise.}\end{cases} \tag{3.5}$$

The above calculations assume a steady velocity field. For a time-dependent incompressible velocity field $v(x,t)$ satisfying the Kolmogoroff assumptions and the same non-dimensionalizations used earlier we obtain (3.4) with the rescaled energy-power spectrum

$$\langle|\hat{v}_\delta(k,\omega)|^2\rangle = \begin{cases} C_0|k|^{1-d-\frac{5}{3}}(|k|^{-\frac{2}{3}}\phi(\omega/|k|^{\frac{2}{3}})), & \delta<|k|<1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.6}$$

and ^ denotes the space-time Fourier transform. Here $\phi \geqslant 0$ is a structure function with the normalization $\int \phi(s)\,\mathrm{d}s = 1$. We remark that the combination $\omega/|k|^{\frac{2}{3}}$ arises as the only combination of frequency and wave number independent of Re and simultaneously consistent with the scaling in (3.3) and the energy spectrum in (3.5). Thus, the basic problem of turbulent transport by velocity fields with Kolmogoroff statistics and no mean flow can be reformulated as (3.4) where $v_\delta(x,t)$ has the energy-power spectrum in (3.6).

In theories for eddy diffusivity, the initial data varies on the integral scale, L_0, and with the above non-dimensionalization has the form

$$T|_{t=0} = T_0(\delta x), \quad \delta \ll 1. \tag{3.7}$$

The goals of a theory of eddy diffusivity for fully developed turbulence is to assess the effect of the motion of the arbitrarily many smaller length scales on the dynamics for T without resolving the effects of the small scales in detail. Here are the first goals of an eddy diffusivity theory for (3.4) with an energy spectrum such as given in (3.5) or (3.6):

(i) Compute an effective time rescaling function, $\rho^2(\delta)$, so that the rescaling ensemble average

$$\bar{T}(x,t) = \lim_{\delta\to 0}\left\langle T^\delta\left(\frac{x}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle \tag{3.8}$$

has a non-trivial limit. For turbulent transport with the velocity statistics in (3.5) or (3.6), $\rho^2(\delta)$ should equal or exceed $\delta^{\frac{2}{3}}$, the large-scale eddy turnover time. Since $\delta = (Re)^{-\frac{3}{4}}$, the limit in (3.8) corresponds to the high Reynolds number limit.

(ii) Compute the effective equation satisfied by $\bar{T}(x,t)$; this is the 'eddy diffusivity' equation because only large-scale fluctuations are involved in this equation.

Why is the problem in (3.4) with velocity statistics in (3.5) or (3.6) difficult? We are interested in a uniformily valid theory as $(Re)\to\infty$ and there are strong infrared divergences in this limit for the velocity spectra in (3.5) and (3.6). This infrared divergence implies that long-range correlations for the velocity field dominate the dynamics in the high Reynolds number limit. For example, the steady velocity field in (3.5) satisfies

$$\lim_{\delta\to 0}\int_{R^d}\frac{\langle|\hat{v}_\delta(k)|^2\rangle}{|k|^2}\,\mathrm{d}k = +\infty, \tag{3.9}$$

while a mathematically rigorous necessary and sufficient condition (Avellaneda & Majda 1989, 1991; Majda 1991, p. 371) for simple 'mean field' diffusion theories with $\rho(\delta) = \delta$ to be valid for steady velocity fields is that

$$\int_{R^d}\frac{\langle|\hat{v}(k)|^2\rangle}{|k|^2}\,\mathrm{d}k < \infty. \tag{3.10}$$

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The situation here for spatial correlations is analogous to that already developed earlier in the simple model in §2*b*, *c* for time correlations where sufficiently long range correlations cause a 'phase transition' and completely different super-diffusive behaviour in the large-scale long-time limit as a spectral parameter crosses from $\epsilon < 0$ to $\epsilon > 0$.

(ii) *Velocity statistics in the model*

For the model problem we embed the analogue of the Kolmogoroff spectrum in (3.6) with $d = 1$ into a two-parameter family of turbulent velocity fields for (3.1) depending on the parameters ϵ, z. The parameter ϵ measures the rate of infrared divergence in space according to the precise criterion in (3.10) while the parameter z measures through $|k|^z$ the correlation length of velocity components with wave number k, the Kolmogoroff spectrum in (3.6) has $z = \frac{2}{3}$. We assume that the model problem in (3.1) has been non-dimensionalized on dissipation length scales in a fashion as described in (3.3)–(3.6) for the Kolmogoroff spectrum and we identify κ in (3.1) with $(Pr)^{-1}$ where Pr is the Prandtl (or Schmidt) number.

We consider stationary mean-zero velocity fields for (3.1). The space-time correlations for the velocity are given in general by the energy-power spectrum through Fourier transform by

$$\langle v(x'+x, t'+t)\, v(x', t')\rangle = R(x,t) = \iint \mathrm{e}^{2\pi \mathrm{i}(kx+\omega t)}\, E(k,\omega)\, \mathrm{d}k\, \mathrm{d}\omega. \tag{3.11}$$

As in (3.6) above, we assume that the velocity statistics for (3.1) have the form

$$E(k,\omega) = \bar{V}^2 |k|^{1-\epsilon} (|k|^{-z} \phi(\omega/|k|^z))\, \psi_0(|k|/\delta)\, \psi_\infty(|k|), \tag{3.12}$$

with ϵ, z satisfying $-\infty < \epsilon < 4$ and $0 \leqslant z \leqslant +\infty$ (the limit $z = +\infty$ for (3.12) corresponds to steady velocity fields). We assume that $\phi \geqslant 0$ is a structure function with $\int \phi = 1$. With non-dimensionalization for (3.1) analogous to those in (3.3) for the Kolmogoroff spectrum, $\psi_\infty(|k|)$ represents the behaviour of the velocity on the dissipation scales while $\psi_0(|k|/\delta)$ is an infrared cut-off at the integral scale $L_0 = \delta^{-1}$ expressing the fact that the velocity field is deterministic and not statistically universal for these larger length scales; we assume that $\psi_\infty(|k|) \geqslant 0$ is rapidly decreasing for large k with $\psi_\infty(0) = 1$ and that $\psi_0(|k|) \geqslant 0$ vanishes in a neighbourhood of zero. Particular examples satisfying these conditions and motivated by (3.5) and (3.6) are

$$\psi_0(|k|) = \begin{cases} 1, & |k| > 1, \\ 0, & |k| \leqslant 1, \end{cases} \quad \text{and} \quad \psi_\infty(|k|) = \begin{cases} 1, & |k| \leqslant 1, \\ 0, & |k| > 1. \end{cases} \tag{3.13}$$

Clearly the analogue in the model of the Kolmogoroff spectrum is given by the values $\epsilon = \frac{8}{3}$ and $z = \frac{2}{3}$ with the cut-offs in (3.13).

With (3.11) and (3.12) we have the formula

$$\langle v(x,t)\, v(0,t')\rangle = \bar{V}^2 \int \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x}\cdot\boldsymbol{k}}\, |k|^{1-\epsilon} \psi_0\!\left(\frac{|k|}{\delta}\right) \psi_\infty(|k|) f(|k|^z\, |t-t'|)\, \mathrm{d}k, \tag{3.14}$$

where $f(t)$ is the Fourier transform of the structure function $\phi(\omega)$. In our earlier work (Avellaneda & Majda 1990*a*, 1992*a*), the special choice, $\phi(\omega) = a^{-1}\pi^{-1}(1+\omega^2/a^2)^{-1}$, was utilized for calculational simplicity with the corresponding f given by

$$f(t) = \mathrm{e}^{-a 2\pi |t|}. \tag{3.15}$$

It should be mentioned here that the large-scale theory for eddy diffusivity and higher-order statistics in that work remains valid for a much wider class of structure functions, $\phi(\omega)$. To understand the role of the parameters, ϵ, z, in a more intuitive fashion we utilize (3.14). From this formula we see that the spatial correlations of the velocity at fixed time for $|x-x'| \gg 1$ are given by

$$\langle v(x,t)\, v(x',t)\rangle \sim \text{const.} \times |x-x'|^{-2+\epsilon} \tag{3.16}$$

in the limit as $\delta \to 0$ so that larger values of ϵ correspond to longer range spatial correlations in the velocity field. In particular, we have chosen ϵ in (3.14) so that the correlation functions at fixed time satisfy (3.10) for all values z provided that $\epsilon < 0$; thus, the spatial correlations of the velocity field decay sufficiently rapidly so that (3.10) is satisfied and simple diffusion (mean field) theories occur. As regards the parameter z, we see from (3.14) with the special choice in (3.15) for $f(t)$ that the decorrelation in time of the velocity field components at wave number k for each k is governed by the k-dependent turnover time

$$\tau_a(k) = 1/a|k|^z.$$

This turnover time is longer for smaller values of $|k|$ and incorporates the feature that long wavelength components have longer turnover times. Thus with $|k| \ll 1$, in the limit as $z \to \infty$, the correlation time becomes infinite at all long wavelengths and we recover steady velocity fields with spatial correlations behaving as in (3.16). In the other extreme limiting case with $z = 0$, all modes have identical correlation times regardless of wavelength. Clearly velocity fields with the analogue of the Kolmogoroff spectrum in the model with $\epsilon = \frac{8}{3}$ and $z = \frac{2}{3}$ exhibit strong infrared divergence in space and relative decorrelation in time of different large-scale modes which is intermediate between these two extreme cases.

In this paper we will use the limiting case of velocity statistics which are gaussian with gaussian white noise in time and satisfy (3.14) as ϵ varies. Thus in this extreme limiting case, we have

$$\langle v(x,t)\, v(x',t')\rangle = V^2 \delta(t-t') \int \mathrm{e}^{2\pi \mathrm{i}(x-x')\,k}\, |k|^{1-\epsilon} \psi_0\!\left(\frac{|k|}{\delta}\right) \psi_\infty(|k|)\, \mathrm{d}k, \tag{3.17}$$

where $\delta(t)$ is the Dirac delta measure. We remark that the stationary gaussian velocity field in (3.17) can be derived from a gaussian velocity field with correlations in (3.14) with $z = 0$ and the special structure function in (3.15) with $\bar{V}^2$ normalized by $\bar{V}^2 = 4\pi a V^2$ in the limit as the parameter $a \nearrow \infty$. Thus, the examples of gaussian velocity fields with the correlations in (3.17) correspond to parameters of the velocity statistics in (3.14) with $z = 0$ and ϵ varying. According to (3.16) these velocity fields exhibit longer-range spatial correlations as the parameter ϵ increases with complete decorrelation in time for velocities at all wave numbers.

(b) *Theory for eddy diffusivity in the model*

In this section, we describe the theory of renormalized eddy diffusivity for the model in (3.1) with velocities having the turbulent statistics with correlations described by (3.12) or (3.14) as the parameters ϵ and z vary. Thus we consider (3.1) with these statistics and the large-scale initial data

$$T|_{t=0} = T_0(\delta x, \delta y), \quad \delta \ll 1 \tag{3.18}$$

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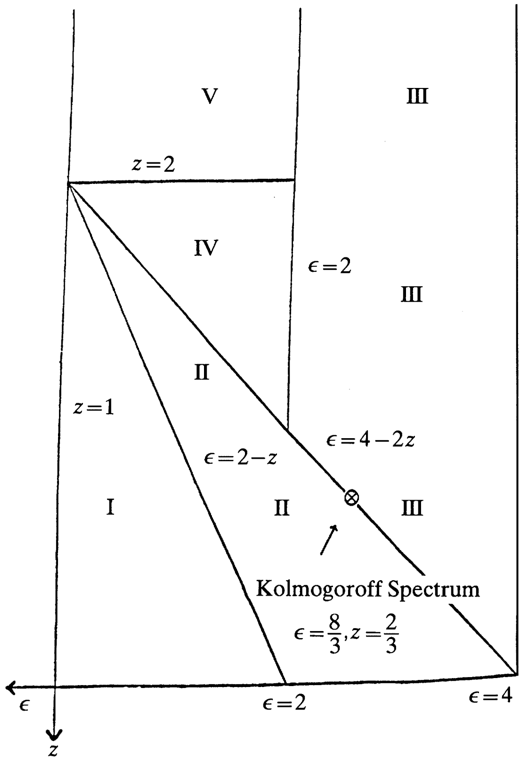

Figure 1. The 'phase diagram' for the five regions in the $(\tilde{\epsilon}, z)$ upper half-plane with different behaviour for the rigorous renormalization theory.

and develop the theory of eddy diffusivity following the objectives mentioned in (3.8). Thus, as in §2, we determine unique large-time rescaling functions

$$x' = \delta x, \quad y' = \delta y, \quad t' = \rho^2(\delta)\, t \tag{3.19}$$

so that with $T(x, y, t)$ the solution of (3.1) with the large-scale initial data in (3.18), the 'high Reynolds number' rescaled limit, $\bar{T}$, defined by

$$\bar{T}(x', y', t') = \lim_{\delta \to 0} \left\langle T\left(\frac{x'}{\delta}, \frac{y'}{\delta}, \frac{t'}{\rho^2(\delta)}\right)\right\rangle \tag{3.20}$$

is non-trivial and $\bar{T}(x, y, t)$ satisfies an effective equation; this is the equation for 'eddy diffusivity' since the effect of all the small scales for the velocity field has been replaced by a different effective equation for the ensemble average involving only the large scales. As in §2*b* (i), the limiting ensemble average for more general initial data than (3.18) can be studied with easy modifications.

The rigorous renormalization theory for the exactly solvable model involves five distinct regions with different scaling laws and a different structure for the effective equations as the boundaries between these regions are crossed (Avellaneda & Majda 1990*a*, 1992*a*). As in the simpler case described in §2*c*, the exponent of $\rho(\delta)$ serves as an order parameter in describing the 'phase transitions' across the boundaries of these five regions; such transitions are described by the 'phase diagram' depicted in figure 1. This phase diagram involves five regions with different anomalous scaling exponents and cross-overs between diffusive, super-diffusive, and super-ballistic

Table 1. *The exact renormalization theory for eddy diffusivity*

region	scaling function	effective equation
I	$\rho(\delta)=\delta$	$\bar{T}_t=\kappa\Delta\bar{T}+D_{\mathrm{I}}\bar{T}_{yy}$, $D_{\mathrm{I}}=2\bar{U}^2\int_0^{\Lambda_0}(2\pi a\lvert k\rvert^z+4\pi^2\kappa\lvert k\rvert^2)^{-1}\lvert k\rvert^{1-\epsilon}\,\mathrm{d}k$
II	$\rho(\delta)=\delta^{(4-\epsilon-z)/2}$	$\bar{T}_t=D_{\mathrm{II}}\bar{T}_{yy}$, $D_{\mathrm{II}}=(\pi a)^{-1}(\epsilon+z-2)^{-1}\bar{U}^2$
III	$\rho(\delta)=\delta^{1-\epsilon/4}$	$\bar{T}_t=tD_{\mathrm{III}}\bar{T}_{yy}$, $D_{\mathrm{III}}=2(\epsilon-2)^{-1}\bar{U}^2$
IV	$\rho(\delta)=\delta^{z/(\epsilon+2z-2)}$	$\bar{T}_t=t^{1+(\epsilon-2)/z}D_{\mathrm{IV}}\bar{T}_{yy}$, $D_{\mathrm{IV}}=(2\pi a)^{(\epsilon-2)/z}2\bar{U}^2\times\left(2+\frac{\epsilon-2}{z}\right)\int_0^{\infty}\lvert k\rvert^{1-\epsilon-z}\left[1-\frac{1-\epsilon^{-\lvert k\rvert^z}}{\lvert k\rvert^z}\right]\mathrm{d}k$
V	$\rho(\delta)=\delta^{1/(1+\epsilon/2)}$	non-local (Avellaneda & Majda 1990*a*)
Kolmogoroff boundary	$\rho(\delta)=\delta^{(4-\epsilon-z)/2}$	$\bar{T}_t=D(t)\bar{T}_{yy}$, $D(t)=(\bar{U}^2/\pi a)\int_1^{\infty}k^{1-\tilde{\epsilon}-z}(1-\mathrm{e}^{-2\pi a k^z t})\,\mathrm{d}k$

scaling régimes with corresponding remarkable changes in the nature of the effective equations for the eddy diffusivity. The structure of the effective equations for $\bar{T}$ for the five regions is presented in table 1 for the explicit cut-offs in (3.13) and the time structure function in (3.15). It is very interesting that the analogue of the Kolmogoroff spectrum with the values $\epsilon=\frac{8}{3}$, $z=\frac{2}{3}$ in the simplified model occurs at a 'phase transition' boundary between two regions with a different renormalization theory for eddy diffusivity (Avellaneda & Majda 1990*a*). We remark that for this Kolmogoroff value, $\rho(\delta)=\delta^{\frac{1}{3}}$, corresponding to the large-scale eddy turnover time in (3.3). The effective equation at this Kolmogoroff boundary is also given in table 1. The significance of this boundary point for the renormalized large-scale higher-order statistics such as the pair distance function and the fractal dimension of interfaces is discussed and developed in detail in recent work (Avellaneda & Majda 1992*a*); the nature of the different effective equations in the phase diagram as well as the sweeping effects of constant mean large-scale motions are also developed in that reference. Below we describe the theory for eddy diffusivity in detail for two instructive limiting cases.

(i) *Turbulent shear velocities and Kac's formula*

Why can we develop an exact renormalization theory in the simplified model? If we take the partial Fourier transform with respect to y,

$$\hat{T}(x,t,\xi)=\int \mathrm{e}^{-2\pi\mathrm{i}y\cdot\xi}\,T(x,y,t)\,\mathrm{d}y,$$

solutions of the model equation in (3.1) satisfy the transformed equation

$$\partial\hat{T}/\partial t+(2\pi\mathrm{i}\xi v(x,t)-\kappa 4\pi^2\xi^2)\,\hat{T}=\kappa\hat{T}_{xx} \tag{3.21}$$

with appropriate initial data from either (3.1) or (3.18) which we omit here. When $\kappa=0$, the equation in (3.21) involves decoupled but strongly correlated oscillators with suitable long-range correlations in x and decorrelation in time described by the parameters ϵ and z. The key idea of the authors in their analysis (Avellaneda & Majda 1990*a*, 1992*a*) is to represent solutions of the non-self-adjoint problem in (3.21) with $\kappa \neq 0$ through Kac's version of the celebrated Feynman–Kac formula (Kac 1950; McKean 1969) as a function space integral and then to do appropriate rigorous asymptotics on this function space integral. We will illustrate these ideas below in the renormalization theory for (3.1) with the velocity statistics in (3.17).

(ii) *Renormalization with steady velocities for* $0 < \epsilon < 2$

The special case of steady velocity fields for (3.1) with long-range correlations is an important simplified model for anomalous diffusion in layered porous media (Dagan 1987). The velocity correlations in this case are given by (3.14) with $f \equiv 1$. The large-scale long-time renormalization theory for (3.1) with steady velocity fields is given by figure 1 and table 1 in the limit as $z \nearrow \infty$; thus, as in §2*c* there is a 'phase transition' from standard diffusion theory for $\epsilon < 0$ to super-diffusion for $0 < \epsilon < 2$. The mean square displacements in this régime were calculated as examples to illustrate super-diffusion in porous media in pioneering work (Matheron & de Marsily 1980); the large-scale long-time renormalized Green's function for (3.1) in this régime was calculated for the first time by the authors (Avellaneda & Majda 1990*a*) by utilizing Kac's formula following the strategy summarized in §3*b*(i). This example displays remarkable effects of diffusion, i.e. $\kappa \neq 0$, after renormalization even though the overall large-scale long-time behaviour is super-diffusive. We discuss this briefly here.

For steady velocity fields and ϵ with $0 < \epsilon < 2$, the renormalized scaling function from (3.19) is given by

$$\rho(\delta) = \delta^{1/(1+\epsilon/2)}. \tag{3.22}$$

With this scaling law, we might guess naively that the effective equation for $\bar{T}$ is a simple local diffusion equation,

$$\partial \bar{T}/\partial t = t^{\frac{\epsilon}{2}} \alpha \mathscr{D} \bar{T}_{yy}, \tag{3.23}$$

where $\mathscr{D}$ depends on κ, i.e.

$$\mathscr{D} = \kappa^{\frac{\epsilon}{2}-1} \tag{3.24}$$

and α has some prescribed value, $\bar{\alpha}(\epsilon)$. The reasoning for this naive guess is the following: (1) solutions of (3.23) remain invariant under the space-time symmetry group associated with the scaling law in (3.22) as described earlier in (2.38); both the explicit dependence on t in (3.23) and the dependence on the bare diffusivity, κ, in (3.24) match the limiting behaviour at $\epsilon = 0, 2$ from the mean field régime for $\epsilon < 0$ and the super-ballistic régime for $\epsilon > 2$.

Nevertheless, this naive guess is completely wrong! Let $K(y, t, \alpha)$ denote the explicit Green's function for (3.23) for a fixed value of α. This Green's function has the explicit kernel

$$K(y,t,\alpha) = (4\pi)^{-\frac{1}{2}} t^{-\frac{1}{2}-\epsilon/4} (\mathscr{D}\alpha)^{-\frac{1}{2}} \exp\{-|y|^2/4D\alpha t^{1+\epsilon/2}\}. \tag{3.25}$$

Then, for each value of ϵ with $0 < \epsilon < 2$ there is a broad band distribution function of diffusivities $\nu_\epsilon(\alpha)$ so that the Green's function for the effective equation is given by

$$K^\epsilon(y,t) = \int_{0^+}^{\infty} K(y,t,\alpha)\, \mathrm{d}\nu_\epsilon(\alpha) \tag{3.26}$$

with the large-scale long-time renormalized mean, $\bar{T}$, from (3.20) given by

$$\bar{T}(x, y, t) = \int K^{\epsilon}(y-\tilde{y}, t)\, T_0(x, \tilde{y})\, \mathrm{d}\tilde{y}. \tag{3.27}$$

In particular, since $\mathrm{d}\nu_{\bar{\epsilon}}(\alpha)$ is not a point mass, the Green's function from (3.26) does not have a gaussian profile for $0 < \epsilon < 2$. The formulas for the distribution function $\nu_{\epsilon}(\alpha)$ for random diffusivity exhibit remarkable changes as the parameter, ϵ, is varied (Avellaneda & Majda 1990*a*).

With (3.27) the equation for eddy diffusivity satisfied by $\bar{T}$ is nonlocal and we can follow the procedure developed by Tartar (1989) to find the structure of this equation. However, unlike the homogenization theory for the steady inviscid shear layers in Tartar's work, these equations would involve a subtle dependence on the bare diffusivity κ as a consequence of the formula in (3.24). It is worth remarking here that despite the special form of the Green's function in (3.27), the same Green's function occurs with renormalization for a large class of non-gaussian velocity statistics (Avellaneda & Majda 1992*a*). Furthermore, a Green's function with the same structure arises after renormalization when the simple steady shear layer for $0 < \epsilon < 2$ is perturbed by an arbitrary periodic incompressible two-dimensional velocity field (Avellaneda & Majda 1992*d*).

(iii) *Eddy diffusivity for transport-diffusion by shear velocities with gaussian white noise in time*

With the discussion from §3*b* (i), the exact solution of (3.1) with general initial data can be represented for a given realization of the velocity statistics through Kac's formula as the function space integral

$$T(x, y, t) = \iint \exp\left[2\pi \mathrm{i}(x\eta + y\xi)\right] \exp\left[-\kappa 4\pi^2 \xi^2 t\right] \mathrm{d}\eta\, \mathrm{d}\xi$$
$$\times E_{\beta}\left\{\exp\left[2\pi \mathrm{i}\eta(2\kappa)^{\frac{1}{2}}\beta(t) - 2\pi \mathrm{i}\xi \int_0^t v(x + (2\kappa)^{\frac{1}{2}}\beta(s), t-s)\,\mathrm{d}s\right]\right\} \hat{T}_0(\eta, \xi). \tag{3.28}$$

Here and below, $\beta(s)$ is a brownian path with $\beta(0) = 0$ and $E_{\beta}[\,\cdot\,]$ denotes the expected value over such paths with respect to Wiener measure (McKean 1969). We calculate that

$$\left\langle T\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle$$
$$= \iint \mathrm{e}^{2\pi \mathrm{i}(x\eta + y\xi)}\, \mathrm{d}n\, \mathrm{d}\xi\, \hat{T}_0(\eta, \xi) \exp\left[-\kappa 4\pi^2 (\delta^2/\rho^2)\, \xi t\right] E_{\beta}\Big\{\exp\left[2\pi \mathrm{i}\eta(2\kappa)^{\frac{1}{2}}\delta\beta(t/\rho^2)\right]$$
$$\times \exp\left[-(4\pi^2/2)\xi^2\delta^2 \int_0^{t/\rho^2}\int_0^{t/\rho^2} R((2\kappa)^{\frac{1}{2}}\beta(s) - \beta(s'), s - s')\,\mathrm{d}s\,\mathrm{d}s'\right]\Big\}, \tag{3.29}$$

where $R(x, t)$ is the velocity correlation function in (3.11). In the formula in (3.29) we have utilized the large-scale initial data in (3.18), commuted the average $E_{\beta}[\,\cdot\,]$ with $\langle\cdot\rangle$, and applied a similar identity for computing exponentials of gaussian random fields as we used earlier in (2.5) from §2 (Avellaneda & Majda 1990*a*). The formulas in (3.28) and (3.29) apply for general mean zero stationary gaussian random velocity fields.

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For the special limiting case of the random fields with gaussian white noise in time and correlations given in (3.17) we compute that

$$\int_0^{t/\rho^2}\int_0^{t/\rho^2} R^\epsilon((2\kappa)^{\frac{1}{2}}(\beta(s)-\beta(s')), s-s')\,\mathrm{d}s\,\mathrm{d}s' = \frac{1}{2}\frac{t}{\rho^2}R^\epsilon(0) \tag{3.30}$$

with
$$R^\epsilon(0) = \bar{V}^2\int_{-\infty}^{\infty}|k|^{1-\epsilon}\psi_0\left(\frac{|k|}{\delta}\right)\psi_\infty(|k|)\,\mathrm{d}k \tag{3.31}$$

for $-\infty<\epsilon<4$. We use the standard identity for brownian motion

$$E_\beta\{\exp[2\pi\mathrm{i}\eta(2\kappa)^{\frac{1}{2}}\delta\beta(t/\rho^2)]\} = \exp[-4\pi^2\eta^2\kappa(\delta^2/\rho^2)\,t]. \tag{3.32}$$

Combining (3.29)–(3.32) for these special velocity statistics, we have

$$\left\langle T\left(\frac{x}{\delta},\frac{y}{\delta},\frac{t}{\rho^2}\right)\right\rangle = \iint \mathrm{e}^{2\pi\mathrm{i}(x\eta+y\xi)}\exp\left[-\kappa 4\pi^2\frac{\delta^2}{\rho^2}(\xi^2+\eta^2)\,t-\pi^2\xi^2\frac{\delta^2}{\rho^2}R^\epsilon(0)\,t\right]\hat{T}_0(\eta,\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{3.33}$$

There is different behaviour in the limit as $\delta\downarrow 0$ depending on whether $R^\epsilon(0)$ from (3.31) converges or diverges as $\delta\downarrow 0$. We observe from (3.31) that $R^\epsilon(0)$ is absolutely convergent for $\epsilon<2$ and divergent for $\epsilon>2$.

The inequalities $-\infty<\epsilon<2$ define the mean field régime for the theory for eddy diffusivity with the velocity statistics in (3.17). In this régime as $\delta\to 0$,

$$R^\epsilon(0)\to\bar{R}^\epsilon = \bar{V}^2 2\int_0^\infty k^{1-\epsilon}\psi_\infty(k)\,\mathrm{d}k<\infty;$$

as in §2*b* above, we have the normal diffusive scaling $\rho(\delta)=\delta$ and with (3.33) the effective equation satisfied by the large-scale long-time mean, $\bar{T}$, with these scalings is given by

$$\partial\bar{T}/\partial t = \kappa\Delta\bar{T}+\tfrac{1}{4}\bar{R}^\epsilon\,\partial^2\bar{T}/\partial y^2,\quad \bar{T}|_{t=0}=T_0(x,y) \tag{3.34}$$

with
$$\rho(\delta)=\delta \tag{3.35}$$

for $0<\epsilon<2$.

For $2<\epsilon<4$, $R^\epsilon(0)$ diverges in the limit as $\delta\downarrow 0$ and as in §2*c*, a super-diffusive scaling is needed to treat this divergence. We compute that

$$\frac{\delta^2}{\rho^2}R^\epsilon(0) = \frac{\delta^{4-\epsilon}}{\rho^2}V^2\int_{-\infty}^{\infty}|k|^{1-\epsilon}\psi_0(|k|)\,\psi_\infty(\delta|k|)\,\mathrm{d}k \tag{3.36}$$

and furthermore, for $\epsilon>2$ the integral

$$\bar{R}^\epsilon = V^2\int_{-\infty}^{\infty}|k|^{1-\epsilon}\psi_0(|k|)\,\mathrm{d}k \tag{3.37}$$

is convergent. With (3.36), (3.37), and (3.33) we see that we need to utilize the super-diffusive scaling where $\rho^2=\delta^{4-\epsilon}$, i.e.

$$\rho(\delta)=\delta^{2-\epsilon/2},\quad 2<\epsilon<4 \tag{3.38}$$

to renormalize this divergence. Since $\delta^2/\rho^2=\delta^{\epsilon-2}\to 0$ as $\delta\to 0$, as in §2*c*, the effects of 'bare diffusion' with coefficient κ are negligible in this régime and from (3.33), we see that with these rescalings the equation for eddy diffusivity for $\bar{T}$ is

$$\partial \bar{T}/\partial t = \tfrac{1}{4}\bar{R}^{\epsilon}\, \partial^2 \bar{T}/\partial y^2, \quad \bar{T}|_{t=0} = T_0(x, y). \tag{3.39}$$

As we discussed earlier below (3.17), these special velocity statistics correspond to the values of (ϵ, z) with $z = 0$ in the phase diagram from figure 1 and table 1. The time rescaling functions $\rho(\delta)$ from (3.35) and (3.38) correspond exactly to those in the general theory listed in table 1 for $z = 0$ with a phase transition at $\epsilon = 2$ from normal diffusion to super-diffusion. Furthermore, the effective equations in (3.35) and (3.39) for $\bar{T}$ in these régimes have the same form as listed in table 1 for regions I and II with $z = 0$. Thus, the theory for eddy diffusivity which we have just presented in detail for these limiting velocity statistics both illustrates and reproduces the general theory of renormalized eddy diffusivity for (3.1) in an instructive limiting case with $z = 0$.

We remark that unlike the renormalized theory presented in §2*c* and discussed below (2.38), the effective equation for $\bar{T}$ in (3.39) for $2 < \epsilon < 4$ is not scale invariant under the space-time symmetry group

$$(x, t) \rightarrow (\lambda x, \lambda^{4-\epsilon} t) \tag{3.40}$$

associated with the renormalized time rescaling law in (3.38). The intuitive reason for this is that the large-scale long-time ensemble average $\bar{T}$ evolves through velocities which contain the most energy. In the region with infrared divergence for $\epsilon > 2$, these large energy velocity components are concentrated at the largest spatial scales and the renormalized diffusivity $\bar{R}^{\epsilon}$ from (3.37) depends upon the infrared cut-off $\psi_0(|k|)$ reflecting the most energetic part of the velocity spectrum. In this fashion, the scale invariance associated with (3.40) is broken. On the other hand, in (2.35) from §2*c*, there is no dependence of diffusivity on an infrared cut-off, and scale invariance as in (2.40) is satisfied. It is worth remarking here that the effective equations for $\bar{T}$ are scale invariant under the appropriate space-time symmetry group for regions I, IV, and V for the eddy diffusivity theory in figure 1 and table 1 but in regions II and III, the equations for $\bar{T}$ depend on the infrared cut-off as illustrated in the special case above and are not scale invariant. However, more complicated higher-order statistics such as the pair-distance function remain scale invariant under the appropriate space-time symmetry group in all regions unlike the mean statistics involving $\bar{T}$ which we discussed here. The above facts are discussed in a recent paper of the authors (Avellaneda & Majda 1992*a*, §5).

The models in (3.1) are anisotropic. Thus in these models, it is natural to consider the possible anisotropic scaling laws

$$x' = g(\delta)\, x, \quad y' = \delta y, \quad t' = \rho^2(\delta)\, t \tag{3.41}$$

with $g(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ so that the limit

$$\bar{T}(x, y, t) = \lim_{\delta \to 0} \left\langle T\left(\frac{x}{g(\delta)}, \frac{y}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle (h(\delta))^{-1} \tag{3.42}$$

exists and $\bar{T}$ satisfies a non-trivial effective equation (Zhang & Glimm 1992). As in remark §2*b*(i), the amplitude scaling function $h(\delta)$ is needed for general initial data which do not necessarily involve only the large scales, for large-scale initial data $h(\delta) \equiv 1$. We illustrate and discuss these effects of anisotropic scaling on $\bar{T}$ for the special velocity fields with statistics in (3.17).

For general initial data $T_0(x, y)$, by following (3.30)–(3.34) with minor changes, we obtain

$$\left\langle T\left(\frac{x}{g(\delta)}, \frac{y}{\delta}, \frac{t}{\rho^2(\delta)}\right)\right\rangle = \delta g(\delta) \iint \mathrm{e}^{2\pi \mathrm{i}(x\eta+y\xi)}$$

$$\times \exp\left[-\kappa 4\pi^2\left(\frac{(g(\delta))^2}{\rho^2}\eta^2+\frac{\delta^2}{\rho^2}\xi^2\right)t-\pi^2\xi^2\frac{\delta^2}{\rho^2}R^{\epsilon}(0)\,t\right]\hat{T}_0(g(\delta)\,\eta, \delta\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi. \quad (3.43)$$

For simplicity in exposition, we assume that $\iint T_0(x, y) = C_0 \neq 0$ so that

$$\lim_{\delta\to 0} \hat{T}_0(g(\delta)\,\eta, \delta\xi) = C_0 \neq 0 \quad (3.44)$$

and then we utilize the obvious choice $h(\delta) = \delta g(\delta)$ in (3.42). In the mean field régime, $-\infty < \epsilon < 2$, the rescaled limit in (3.42) is non-trivial if and only if

$$\rho(\delta) = \delta \quad \text{and} \quad g(\delta) = \delta^{\theta}, \quad \theta \geqslant 1.$$

When $\theta = 1$, $\bar{T}$ is a constant multiple of the Green's function for (3.34). On the other hand for $\theta > 1$, $\bar{T}$ satisfies the equation

$$\partial\bar{T}/\partial t = (\kappa + \tfrac{1}{4}\bar{R}^{\epsilon})\,\partial^2\bar{T}/\partial y^2, \quad \bar{T}|_{t=0} = C_0\,\delta(x)\,\delta(y). \quad (3.45)$$

In the régime with renormalization, $2 < \epsilon < 4$, with the formula in (3.43) the rescaled limit in (3.42) is non-trivial if and only if

$$\rho(\delta) = \delta^{2-\epsilon/2}, \quad g(\delta) = \delta^{\theta}, \quad \theta \geqslant 2-\tfrac{1}{2}\epsilon. \quad (3.46)$$

For $\theta > 2-\frac{1}{2}\epsilon$, $\bar{T}$ is a multiple of the Green's function for the isotropic renormalized equation for eddy diffusivity in (3.39). On the other hand, with the special anisotropic scaling $\theta = 2-\frac{1}{2}\epsilon$ so that $\rho(\delta) = g(\delta)$, from (3.43) the effective equation for $\bar{T}$ is given by

$$\frac{\partial\bar{T}}{\partial t} = \kappa\frac{\partial^2}{\partial x^2}\bar{T} + \tfrac{1}{4}\bar{R}^{\epsilon}\frac{\partial^2\bar{T}}{\partial y^2}, \quad \bar{T}|_{t=0} = C_0\,\delta(x)\,\delta(y) \quad (3.47)$$

with $\bar{R}^{\epsilon}$ given in (3.37) for $2 < \epsilon < 4$. Zhang and Glimm emphasize the special choice of anisotropic scalings $g(\delta) = \rho(\delta)$ in their work and repeat the authors' calculations for (3.1) in the isotropic case, $g(\delta) = \delta$ described earlier (Avellaneda & Majda 1990*a*, 1992*a*). The authors are genuinely puzzled by the claims in that work (Zhang & Glimm 1992) that the inertial range is studied by such trivial modifications of the large-scale behaviour of the mean statistical quantities, $\bar{T}$. It is well-known and standard in the turbulence community (Lesieur 1990; McComb 1990) that the scaling properties of the inertial range are determined by second-order statistics and not only those involving mean quantities such as $\bar{T}$. Such higher-order statistics are addressed next in §3*c* and at large scales in recent work of the authors (Avellaneda & Majda 1992*a*).

(*c*) *Second-order statistics and the pair distance equation*

Here we study the behaviour of the second-order statistics, $\langle T(x, y, t)\, T(\tilde{x}, \tilde{y}, t)\rangle$ in the simplified model from (3.1). In general, the scaling behaviour of the second-order correlations at high Reynolds numbers in appropriate régimes is directly related to the scaling laws for the inertial range in turbulent transport (Lesieur 1990, ch. 8). When diffusion is neglected so that $\kappa = 0$, it is well-known (Lesieur 1990, p. 233) that

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this second-order statistical quantity is essentially the pair distance function. In this section, we discuss the behaviour of the second-order correlations at arbitrary scales and not only in the large-scale limit as we have done previously (Avellaneda & Majda 1992*a*, §5).

For simplicity in exposition we consider real-valued stationary mean zero gaussian random initial data for (3.1) which is x independent, i.e.

$$T_0(y) = \int e^{2\pi i y\cdot\xi}\,\hat{T}_0(\xi)\,dW(\xi), \tag{3.48}$$

where $dW(\xi)$ is gaussian white noise with $\langle dW(\xi)\,dW(\xi')\rangle = \delta(\xi+\xi')\,d\xi$ and $\hat{T}_0(-\xi) = \hat{T}_0(\xi)$. We utilize Kac's formula from §3*b* (i) as illustrated above in (3.28), (3.29) to give a representation formula for the second-order statistics for (3.1) involving function space integrals. Then we discuss the properties of these higher-order statistics which can be understood through these formulas. In our calculations below, we assume initially only that the velocity field is mean zero and gaussian with correlation function given in (3.11). Later we will specialize to the velocity statistics in the limiting case in (3.17).

By using Kac's formula and similar but more lengthy calculations as in (3.28), (3.29) we obtain the representation formula for the second-order statistics of the solution with the initial data in (3.48)

$$\langle T(x',y',t)\,T(x+x',y+y',t)\rangle = \int e^{2\pi i y\xi}\,|\hat{T}_0(\xi)|^2\,e^{-2\kappa 4\pi^2\xi^2 t}\,E_{\beta_1\beta_2}[\exp(-4\pi^2\xi^2 W(\beta_1,\beta_2,x,t))]\,d\xi. \tag{3.49}$$

Here the function $W(\beta_1,\beta_2,x,t)$ is defined through the recipe

$$\exp(-4\pi^2\xi^2 W(\beta_1,\beta_2,x,t)) = \left\langle \exp\left[-2\pi i\xi\int_0^t [v((2\kappa)^{\frac{1}{2}}\beta_1(s),t-s) - v(x+(2\kappa)^{\frac{1}{2}}\beta_2(s),t-s)]\,ds\right]\right\rangle. \tag{3.50}$$

In (3.49), $E_{\beta_1,\beta_2}[\cdot]$ denotes the expected value over independent brownian paths β_1, β_2 with respect to the product Wiener measure (Avellaneda & Majda 1990*a*, §6, 1992*a*, §5). As a consequence of the special choice of initial data in (3.48), the quantity

$$\langle T(x',y',t)\,T(x+x',y+y',t)\rangle = P(x,y,t) \tag{3.51}$$

is stationary and essentially the pair-distance function when $\kappa = 0$. Of course in (3.49) and (3.51), the average $\langle\cdot\rangle$ denotes ensemble average over both the velocity statistics and the random initial data. The right-hand side of (3.50) determines $W(\beta_1,\beta_2,x,t)$ in the form specified in (3.50) as a consequence of our assumption of gaussian statistics for the velocity field as utilized earlier in (3.29) and (2.5) in §2. In fact, through these formulas we have

$$W(\beta_1,\beta_2,x,t) = \frac{1}{2}\int_0^t\int_0^t [R((2\kappa)^{\frac{1}{2}}(\beta_1(s)-\beta_1(s')),(s-s')) + R((2\kappa)^{\frac{1}{2}}(\beta_2(s)-\beta_2(s')),s-s') - R(x+(2\kappa)^{\frac{1}{2}}(\beta_2(s)-\beta_1(s')),(s-s')) - R(-x+(2\kappa)^{\frac{1}{2}}(\beta_1(s)-\beta_2(s')),s-s')]\,ds\,ds' \tag{3.52}$$

with $R(x,t) = \langle v(x,t), v(0,0)\rangle$ the velocity correlation function from (3.11).

(i) *The pair distance equation with* $\kappa = 0$

If we neglect the diffusion in (3.1) and look at the motion of a passively transported quantity without diffusion, i.e. $\kappa = 0$ or $Pr = +\infty$, the general formulas in (3.49) and (3.52) simplify dramatically. With $\kappa = 0$ in (3.52) we have

$$W(0,0,x,t) = \frac{1}{2}\int_0^t \int_0^t \langle (v(x,s) - v(0,s'))^2 \rangle \, \mathrm{d}s \, \mathrm{d}s' \tag{3.53}$$

so that from (3.49) we obtain that the second order correlations $P(x,y,t)$ satisfy the local diffusion equation

$$\partial P/\partial t = \mathscr{D}_*(t,x)\, \partial^2 P/\partial y^2, \quad P|_{t=0} = \langle T_0(y)\, T_0(0) \rangle \tag{3.54}$$

with non-negative diffusion coefficient $\mathscr{D}_*(t,x)$ given by

$$\mathscr{D}_*(t,x) = \int_0^t \langle (v(x,s) - v(0,0))^2 \rangle \, \mathrm{d}s. \tag{3.55}$$

For the special case of velocity statistics which are gaussian white noise in time, we calculate that

$$\mathscr{D}_*(x) = \tfrac{1}{4}\langle (v(x,0) - v(0,0))^2 \rangle. \tag{3.56}$$

With the velocity statistics in (3.17) in the 'high Reynolds number limit', i.e. as $\delta \to 0$, we have the formula

$$\langle (v^\epsilon(x,0) - v^\epsilon(0,0))^2 \rangle = 2V^2 \int_{-\infty}^{\infty} (1 - \cos(kx))\, |k|^{1-\epsilon} \psi_\infty(|k|)\, \mathrm{d}k \tag{3.57}$$

and the integral converges for all ϵ with $-\infty < \epsilon < 4$. We recall that the model in (3.1) has 'non-dimensionalization' motivated by utilizing the dissipation length scale in fully developed turbulence in (3.3), (3.4) so that bounded x corresponds to dissipation lengths while $x \to \infty$ corresponds to the 'inertial range' in the model for fixed ϵ and z. We claim that the diffusion coefficients $\mathscr{D}_*^\epsilon(x)$ behave in a completely different and discontinuous fashion on the two sides of the phase transition boundary where $\epsilon = 2$ for the special statistics in (3.17).

For $\epsilon < 2$, $|k|^{1-\epsilon}\psi_\infty(|k|)$ is integrable so by the Riemann–Lebesgue lemma

$$\int_{-\infty}^{\infty} \cos(kx)\, |k|^{1-\epsilon} \psi_\infty(|k|)\, \mathrm{d}k \to 0 \quad \text{as} \quad |x| \to \infty$$

and

$$\mathscr{D}_*^\epsilon(x) \to \tfrac{1}{2}\bar{R}^\epsilon \quad \text{for} \quad |x| \to \infty. \tag{3.58}$$

We see that this diffusion coefficient for pair dispersion for large x is twice the large-scale diffusion coefficient computed in (3.34), and particles are asymptotically independent for x in the 'inertial range' for $\epsilon < 2$.

In contrast, pairs of particles are strongly correlated in the inertial range for $\epsilon > 2$. By rescaling (3.57), we obtain that

$$\mathscr{D}_*^\epsilon(x) = |x|^{\epsilon-2} C_\epsilon + o(|x|^{\epsilon-2}) \tag{3.59}$$

for $|x| \to \infty$ with

$$C_\epsilon = V^2 \int_0^\infty (1 - \cos(k))\, k^{1-\epsilon}\, \mathrm{d}k, \quad 2 < \epsilon < 4. \tag{3.60}$$

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Clearly, the constant C_ϵ is finite only for $2 < \epsilon < 4$. Thus the pair-diffusion coefficient $\mathscr{D}^\epsilon_*(x)$ grows like $|x|^{\epsilon-2}$ as a result of the strong correlations in the inertial range for $\epsilon > 2$ but approaches a constant for $\epsilon < 2$.

This discontinuous behaviour of the coefficients for pair dispersion across phase transition boundaries which we have just illustrated is typical of the behaviour of pair-distance functions renormalized at large scales and long times for the model in (3.1) with $\kappa \neq 0$ across the general phase transition boundaries given in figure 1 (Avellaneda & Majda 1992*a*, §5). In particular (3.54)–(3.60) recover this large-scale limiting behaviour for $z = 0$ and ϵ varying. It should be clear to the reader that all of the ideas which we have just presented in the special limiting case apply with suitable modification to the more statistics in (3.14). For the model in (3.1) involving simple shear layers, the eulerian and lagrangian velocity correlation functions coincide and the general formula for $W(0,0,x,t)$ in (3.53) is the familiar expression for pair-dispersion of particles initially separated by a distance x through their lagrangian velocity correlations (McComb 1990; Lesieur 1990).

(ii) *The pair distance equation with $\kappa \neq 0$ and white noise velocity statistics in time*

We investigate briefly the second-order correlations in (3.49) with finite diffusion, $\kappa \neq 0$, for the special velocity statistics with the form in (3.17). With (3.17) we calculate from (3.52) that $W(\beta_1, \beta_2, x, t)$ is given by the simpler formula

$$W(\beta_1,\beta_2,x,t) = \frac{1}{2}\int_0^t [R(0) - R(x + (2\kappa)^{\frac{1}{2}}(\beta_2(s) - \beta_1(s)))]\,\mathrm{d}s. \tag{3.61}$$

Thus if we introduce the 'rotated' brownian motion $\beta = \frac{1}{\surd 2}(\beta_2 - \beta_1)$, we obtain the identity

$$\begin{aligned} &E_{\beta_1,\beta_2}[\exp(-4\pi^2\xi^2 W(\beta_1,\beta_2,x,t)] \\ &\quad = E_\beta\left\{\exp\left[-2\pi^2\xi^2\int_0^t [R(0) - R(x + 2\kappa^{\frac{1}{2}}\beta(s))]\,\mathrm{d}s\right]\right\} \equiv \hat{P}_0(x,\xi,t) \end{aligned} \tag{3.62}$$

for the special velocity statistics in (3.17).

We recognize the second formula in (3.62) as Kac's formula (Kac 1950) for an appropriate real potential. To see this we define a potential $V(x)$ by

$$V(x) = 2\pi^2\xi^2[R(0) - R(x)]. \tag{3.63}$$

Then by the Feynman–Kac formula (McKean 1969) $\hat{P}_0(x,\xi,t)$ from (3.62) satisfies the differential equation,

$$\left.\begin{aligned} &\frac{\partial \hat{P}_0(x,\xi,t)}{\partial t} = 2\kappa\frac{\partial^2}{\partial x^2}\hat{P}_0(x,\xi,t) - 2\pi^2\xi^2[R(0) - R(x)]\,\hat{P}_0(x,\xi,t), \\ &\hat{P}_0(x,\xi,t)|_{t=0} = 1. \end{aligned}\right\} \tag{3.64}$$

With (3.49) and (3.64), we recognize that the second-order correlations

$$\langle T(x',y',t)\,T(x+x',y+y',t)\rangle = P(x,y,t)$$

solve the local diffusion equation,

$$\partial P/\partial t = 2\kappa\,\Delta P + \tfrac{1}{2}[R(0) - R(x)]\,\partial^2 P/\partial y^2, \quad P|_{t=0} = \langle T_0(y)\,T_0(0)\rangle. \tag{3.65}$$

It is worth emphasizing here that the differential equation which we just derived for the second-order correlations is exact without any approximations for the special velocity statistics in (3.17). The white noise statistics in time yield markovian behaviour for the second-order statistics at all scales; in general there are non-markovian effects in the formulas for the second-order statistics in (3.49) and markovian behaviour is recovered only at suitable large scales and long times (Avellaneda & Majda 1992*a*, §5).

In the high Reynolds number limit, i.e. $\delta \to 0$ for fixed values of the Prandtl number, the equation for P in (3.65) becomes

$$\partial P/\partial t = 2\kappa\,\Delta P + \mathscr{D}^{\epsilon}_{*}(x)\,\partial^2 P/\partial y^2, \quad P|_{t=0} = \langle T_0(y)\,T_0(0)\rangle \tag{3.66}$$

with $\mathscr{D}^{\epsilon}_{*}(x)$ calculated earlier in (3.56) and (3.57). As in (3.58)–(3.60), the behaviour of solutions of the differential equation in (3.66) is completely different as the parameter ϵ crosses the phase transition boundary from $\epsilon < 2$ to $\epsilon > 2$. To see this we look at the related problem from (3.64) for $\hat{P}(x, \xi, t)$ given by

$$\frac{\partial \hat{P}}{\partial t} = 2\kappa \frac{\partial^2}{\partial x^2}\hat{P}(x,\xi,t) - (8\pi^2\xi^2\kappa + 4\pi^2\xi^2\mathscr{D}^{\epsilon}_{*}(x))\,\hat{P}(x,\xi,t). \tag{3.67}$$

The right-hand side of (3.67) is essentially a self-adjoint Schrödinger operator with a repulsive potential defined by $\mathscr{D}^{\epsilon}_{*}(x)$. For ϵ with $\epsilon < 2$, from (3.58) we see that $\mathscr{D}^{\epsilon}_{*}(x)$ approaches a constant for large x and this Schrödinger operator has a standard continuous spectrum which involves small perturbations of the free space Schrödinger operator shifted by a constant. In this régime with $\epsilon < 2$ the behaviour for (3.66) at large distances is well approximated by a gaussian profile shape. In contrast, for $2 < \epsilon < 4$, it follows from (3.59) and (3.60) that the repulsive potential $\mathscr{D}^{\epsilon}_{*}(x)$ satisfies

$$\mathscr{D}^{\epsilon}_{*}(x) \to \infty \quad \text{as} \quad |x| \to \infty.$$

In this case, it is well known (Titchmarsh 1946) that the Schrödinger operator in (3.67) has only pure point spectrum (like the harmonic oscillator) with localized proper eigenfunctions and no continuous spectrum. Thus, the function $P(x, \xi, t)$ is expanded in terms of the complete orthonormal basis consisting of the eigenfunctions for (3.67). Clearly for ϵ with $2 < \epsilon < 4$, solutions of the diffusion operator in (3.67) are not well approximated by a simple gaussian profile shape at large distances. Of course, in the limit $\kappa \to 0$ the equation in (3.66) reduces to the pair-distance equation discussed in (3.54)–(3.60).

In recent work, one of the authors (Majda 1993*b*) has been studying the complete dynamic renormalization group involving all higher-order correlation functions for the inertial range in the model from (3.1) with the statistics in (3.17). The explicit renormalized statistical fixed point for the inertial range also demonstrates an explicit and rigorous link between intermediate asymptotics (Barenblatt 1979) and statistical renormalization in the simplest models for turbulence. Other aspects of these models have been used recently (Majda 1993*a*; Fefferman & Majda 1994) to demonstrate the large-scale intermittency that occurs through broader than gaussian probability distributions in turbulent diffusion. The interested reader can find many references to recent experimental and numerical work on these issues in the bibliography of Majda (1993*a*).

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